An estimate of topological pressure for certain random walks on Cayley graphs (Integrated Research on the Theory of Random Dynamical Systems)

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An estimate of topological pressure for certain random walks on Cayley graphs

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0 Introduction

One motivation for this research is Bowen's formula ([Bow79], see also [Bar08] for a recent survey). The formula developed a relation between the fractal dimension s of a hyperbolic limit set, consisting of accumulation points of orbits of a group action, and the root of an associated topological pressure function $\mathcal{P}$. Moreover, the accumulation points of orbits generate random walks. In particular, if the group is free, then we may consider the Non-Backtracking Random Walk (for short NBRW). Roughly speaking, a NBRW is a random walk that is not allowed to go backwards at each step except the first one. If we know how the pressure changes when passing from a free group to a quotient group, then we are able to estimate the difference of dimensions of the corresponding limit sets. In this paper, the NBRWs generated by the free group and its quotients are modeled by the Topological Markov Shifts (Σ_A, θ) and $(\widetilde{\Sigma}_A, T)$, respectively.

In [OW07], cogrowth, spectral radius of transition matrix and amenability have been investigated by NBRWs. Even for simple random walks, we know that these three notions are deeply related ([GdlH97]). The (Gurevičh) pressure of a potential φ on (Σ_A, θ) is denoted by $\mathcal{P}(\Sigma_A, \theta, \varphi)$ and can be considered as a generalization from simple random walks to weighted random walks. For such φ, there is a natural way to extend φ to $\widetilde{\Sigma}$. We will also use φ to denote this extended potential, the pressure is denoted by $\mathcal{P}(\widetilde{\Sigma}, T, \varphi)$. We will assume that φ is normalized, hence in particular $\mathcal{P}(\Sigma_A, \theta, \varphi) = 0$. See the beginnings of Section 2 and [Sar] for the details.

In this setting, it is a natural task to estimate the pressure of quotients of the free group. In order to estimate, we make use of the analog of Cheeger's isoperimetric inequalities ([Che70], [Moh88] and a weighted version [Woe00]). We will use a method in the context of isoperimetric inequalities, and combine this method with the previous research of Stadlbauer [Sta13]. Specifically, for $n \geq 2$ and a non-amenable group $\mathbb{F}_n/N$, where $\mathbb{F}_n$ is a free group generated by n free generators and N is a normal subgroup of $\mathbb{F}_n$, and a potential φ, our main result (see Theorem 2.0.8) states that there exists positive α and δ where δ is derived by the above method, and $k \in \mathbb{N}$ such that we have

$$\mathcal{P}(\widetilde{\Sigma}_A, T, \varphi) \leq \frac{1}{k} \log(1 - \alpha\delta)$$

where k depends only on the length ℓ of the shortest word length of N, α depends only on φ, and δ depends on $\mathbb{F}_n/N$.

1 Preliminaries

1.1 Topological Markov shift

For a finite or countable set I, let $A = [a_{ij}]_{I \times I}$ be a matrix of zeros and ones with no columns or rows which are all zeros. For simplicity, we denote $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$.

Definiton (Topological Markov Shift (TMS), Cylinder set, Shift map)
The *topological Markov shift* Σ_A with set of states I and transition matrix A is the set

$$\Sigma_A := \{(i_n) \in I^{\mathbb{N}_0} \mid a_{i_n i_{n+1}} = 1, \ {}^\forall n \in \mathbb{N}_0\},$$

equipped with the topology generated by the collection of *cylinders* of length m

$$[(w_0, \dots w_{m-1})] := \{(i_n) \in \Sigma_A \mid i_n = w_n, \ 0 \leq n \leq m\}$$

for all $m \in \mathbb{N}$ and $w_0, \dots, w_m \in I$ and endowed with the action of the *left shift* map

$$\theta : \Sigma_A \to \Sigma_A; (i_0, i_1, i_2, \dots) \mapsto (i_1, i_2, i_3, \dots).$$

A *word* is an element $(i_0, \dots, i_{n-1}) \in I^n$ $(n \in \mathbb{N})$. The length of the word is n. A word is called *admissible* (with respect to a transition matrix A) if the cylinder set generated by that word is not empty. Let us denote the set of all admissible words of length n by $\mathcal{W}^n$ and of all admissible word by $\mathcal{W}^\infty$, that is, $\mathcal{W}^\infty = \bigcup_{k=1}^\infty \mathcal{W}^k$.

1.2 Extension by groups

Let (Σ_A, θ) be a TMS and G be a countable group.

Definiton (G-extension)
$(\widetilde{\Sigma}_A, T)$ is called a $G-$*extension* of (Σ_A, θ)
$\Leftrightarrow {}^\exists \psi : \Sigma_A \to G$ depending only on the first coordinate of $(i_0, i_1, \dots)$ such that $\widetilde{\Sigma}_A := \Sigma_A \times G$ and $T : \widetilde{\Sigma}_A \to \widetilde{\Sigma}_A$ is for $((i_0, i_1, \dots), g) \in \widetilde{\Sigma}_A$ given by ,

$$T((i_0, i_1, \dots), g) = ((i_1, i_2, \dots), g\psi((i_0, i_1, \dots))).$$

Note that $(\widetilde{\Sigma}_A, T) = (\Sigma_A \times G, T)$ is a skew product over (Σ_A, θ),

$$\begin{array}{ccc} \Sigma_A \times G & \xrightarrow{T} & \Sigma_A \times G \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \Sigma_A & \xrightarrow{\theta} & \Sigma_A \end{array}$$

and $(\widetilde{\Sigma}_A, T)$ is also a TMS with countable state space $(I \times G)$ and that its cylinder sets are given by $[w, g] := [w] \times \{g\}$, for $w \in \mathcal{W}^\infty$..

For potential function $\varphi : \Sigma_A \to \mathbb{R}$, there is a natural extended potential on $\widetilde{\Sigma}_A$ such that $(w, g) \mapsto \varphi(w)$. We also use φ to denote the extended potential. In addition, we define Φ_n as

$$\Phi_n(x) := \prod_{k=0}^{n-1} \varphi(T^k x), \tag{1.2.1}$$

for $x \in \widetilde{\Sigma}_A$.

The *Gurevič pressure* $\mathcal{P}(\widetilde{\Sigma}_A, T, \varphi)$ is defined as the exponential growth rate of

$$\mathcal{Z}^n_{w,g} := \sum_{\substack{y \in [x] = [w,g] \\ T^n y = y}} \Phi_n(y)$$

for a fixed $w \in I = \mathcal{W}^1 = \mathcal{W}$ and $g \in G$. That is,

$$\mathcal{P}(\widetilde{\Sigma}_A, T, \varphi) := \limsup_{n \to \infty} \log \sqrt[n]{\mathcal{Z}^n_a} = \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{Z}^n_a.$$

In fact, this formula does not depend on the choice of $a \in I \times G$ if $(\Sigma_A \times G, T)$ is topologically transitive. Throughout this paper, $(\Sigma_A \times G, T)$ will always be topologically transitive.

For $v \in \mathcal{W}^\infty$, the inverse branch given by $[v, \cdot]$ will be denoted by τ_v, that is, if v is a length n, then $\tau_v : T^n[v, \cdot] \to [v, \cdot]; (x, g) \mapsto (vx, g\psi(v)^{-1})$.

For a function f on Σ_A (resp. $\widetilde{\Sigma}_A$), the *Ruelle operator* L_φ (resp. $\mathcal{L}_\varphi$) with respect to θ (resp. T) and a potential φ is defined as follows. For $\xi \in \Sigma_A$

$$L_\varphi(f)(\xi) := \sum_{v \in \mathcal{W}} (\varphi \circ \tau_v)(\xi) \cdot (f \circ \tau_v)(\xi),$$

resp. for $\xi \in \Sigma_A$ and $g \in G$

$$\mathcal{L}_\varphi(f)(\xi, g) := \sum_{v \in \mathcal{W}} (\varphi \circ \tau_v)(\xi) \cdot (f \circ \tau_v)(\xi, g).$$

2 Proofs of results

From here, let G be a group generated by two generators $\{g_1, g_2\}$, $I = \{\pm 1, \pm 2\}$ and $A = [a_{ij}]_{I \times I}$ such that $a_{ij} = 0$ whenever $i = -j$ and $a_{ij}1$ otherwise. Put $g_{-i} = g_i^{-1}$

for $i = 1, 2$. G can be represented by the quotient group $\mathbb{F}_2/N$, where $\mathbb{F}_2$ is a free group generated by two free generators and N is a normal subgroup of $\mathbb{F}_2$. Then we can easily check that if the function ψ on $I = \mathcal{W}$ is defined as $\psi(i) = g_i$, then (Σ_A, θ) is topologically mixing and its extension $(\widetilde{\Sigma}_A, T)$ is topologically transitive.

The above settings may be considered a Non-Backtracking Random Walk on the Cayley graph of G. A further important consequence of topologically mixing and finite alphabet is the existence of an invariant Gibbs measure. That is, if (Σ_A, θ) is topologically mixing and a finite alphabet, $\log \varphi$ is Hölder continuous and $\|L_\varphi \mathbf{1}\|_\infty < \infty$, then there exist a Gibbs measure μ for φ and a positive Hölder-continuous eigenfunction h of L_φ such that $h \cdot d\mu$ is an invariant probability measure. By replacing φ by $\varphi + \log h - \log(h \circ \theta)$, we may assume from now on that $L_\varphi \mathbf{1} = \mathbf{1}$ and $\mathcal{P}(\Sigma_A, \theta, \varphi) = 0$.

The existence of μ then gives rise to the following definition of $\mathcal{H}_\infty$. Given a measurable function $f : \widetilde{\Sigma}_A \to \mathbb{R}$, $g \in G$, set $\|f\|_1^g := \|f(\cdot, g)\|_1 = \int_{w \in \Sigma_A} |f(w, g)| d\mu(w)$ and define

$$[\![f]\!]_1 := \sqrt{\sum_{g \in G} \{\|f\|_1^g\}^2} \text{ and } \mathcal{H}_\infty := \{f : \widetilde{\Sigma}_A \to \mathbb{R} \mid [\![f]\!]_1 < \infty\}.$$

Furthermore, set $\mathcal{H}_c := \{f \in \mathcal{H}_\infty \mid f \text{ is constant on } \Sigma_A \times \{g\}, {}^\forall g \in G\}$. If $f \in \mathcal{H}_c$, then we define $\hat{f}(g) := f(x, g)$ for any $g \in G$ and $x \in \Sigma_A$ since f does not depend on the first coordinate, and then we have for all $g \in G$

$$\|f\|_1^g = \sum_{i \in \mathcal{W}} \hat{f}(g) \mu([i]) = \hat{f}(g),$$

and this implies that

$$[\![f]\!]_1 = \sqrt{\sum_{g \in G} \{\|f\|_1^g\}^2} = \sqrt{\sum_{g \in G} \hat{f}^2(g)} = \|\hat{f}\|_{\ell^2(G)}. \tag{2.0.1}$$

Recall that a Banach space $(B, \|\cdot\|)$ is *uniformly convex* (also called *uniformly rotund*) if for all $\delta > 0$ there exists $\varepsilon > 0$ such that, for all f, g with $\|f - g\| \geq \delta$ and $\|f\| = \|g\| = 1$, it follows that $\|f + g\| \leq 2 - \varepsilon$. In particular, $\mathcal{H}_c$ has this property because any Hilbert space H equipped with a norm $\|*\|$ satisfies parallelogram law, that is, for every $f, g \in H$ we have

$$\|f + g\|^2 + \|f - g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right).$$

This implies, for every $f, g \in \mathcal{H}_c$ (closed subspace of $\mathcal{H}_1$) with $[\![f - g]\!]_1 = \|\hat{f} - \hat{g}\|_{\ell^2(G)} \geq \delta$,

$$\begin{aligned} \|\hat{f} + \hat{g}\|_{\ell^2(G)} &= [\![f + g]\!]_1 = \sqrt{2\left([\![f]\!]_1^2 + [\![g]\!]_1^2\right) - [\![f - g]\!]_1^2} \\ &\leq \sqrt{2\left([\![f]\!]_1^2 + [\![g]\!]_1^2\right) - \delta^2} = \sqrt{2\left(\|\hat{f}\|_{\ell^2(G)}^2 + \|\hat{g}\|_{\ell^2(G)}^2\right) - \delta^2}. \end{aligned}$$

If furthermore $\|\hat{f}\|_{\ell^2(G)} = \|\hat{g}\|_{\ell^2(G)} = c$, then

$$\begin{aligned} \|\hat{f} + \hat{g}\|_{\ell^2(G)} &= c\|\hat{f}/c + \hat{g}/c\|_{\ell^2(G)} \\ &\leq c\sqrt{4 - (\delta/c)^2}. \end{aligned} \tag{2.0.2}$$

Lemma 2.0.1
Suppose that N has an element of word length $\ell \geq 1$. Then there exists a finite subset $\mathcal{J}$ of $\mathcal{W}^{\ell+4}$ such that for each pair (β, β') with $\beta, \beta' \in I$ there exists $w_{\beta,\beta'} \in \mathcal{J}$ such that $\beta w_{\beta,\beta'} \beta'$ is admissible and $\psi_\ell(w_{\beta,\beta'}) = e$, where e is a unit element of G.

Remark 2.0.2
If the $G = \mathbb{F}_n/N$ with $n \geq 3$, then we can replace $\ell + 4$ by $\ell + 2$ in Lemma 2.0.1.

Definiton 2.0.3
Let $M_\ell(j) = \mu([j]) + \mu([\widehat{j}])$ for every $\ell \in \mathbb{N}$ and $j \in \mathcal{W}^\ell$. Then

$$\mathfrak{i}(M_\ell) := \inf_{\substack{X \subset G \\ X\,:\,\text{finite}}} \frac{M_\ell(\partial X)}{|X|},$$

where

$$M_\ell(\partial X) := \sum_{*} M_\ell(j)\,, \quad \text{where } * \text{ denote } j \in \mathcal{W}^\ell \text{ s.t. } {}^\exists g \in X, {}^\exists h \in X^c \text{ and } g g_j = h.$$

Let $\mathfrak{i}_{S,\ell}$ be a (standard) isoperimetric constant in terms of equidistribution, that is, it is the case that $M_\ell(j) = 1/|\mathcal{W}^\ell|$ for any $j \in \mathcal{W}^\ell$.

Lemma 2.0.4
For every $\ell \in \mathbb{N}$, $\mathfrak{i}_{S,\ell} = 0$ if and only if $\mathfrak{i}(M_\ell) = 0$.

Proof. Put $M = \max_{j \in \mathcal{W}^\ell} M(j)$ and $m = \min_{j \in \mathcal{W}^\ell} M(j)$. Since M and m are not 0, we can take constants C_1 and C_2 such that $\frac{C_1}{|\mathcal{W}^\ell|} \leq m$ and $M \leq \frac{C_2}{|\mathcal{W}^\ell|}$, respectively. It implies that for any finite subset X of G, we have

$$C_1 \sum_{*} \frac{1}{|\mathcal{W}^\ell|} \leq \sum_{*} m \leq M_\ell(\partial X) = \sum_{*} M_\ell(j) \leq \sum_{*} M \leq C_2 \sum_{*} \frac{1}{|\mathcal{W}^\ell|}.$$

We therefore conclude that

$$C_1 \cdot \mathfrak{i}_{S,\ell} \leq \mathfrak{i}(M_\ell) \leq C_2 \cdot \mathfrak{i}_{S,\ell}.$$

□

Remark 2.0.5
By [Woe00, Proposition 12.4], a group G is non-amenable if and only if $\mathfrak{i}_{S,\ell} \neq 0$ on the Cayley graph generated by G for every $\ell \in \mathbb{N}$. From here, we will therefore suppose that the group G is non-amenable. And given M_ℓ, we denote isoperimetric number in terms of M_ℓ by $\mathfrak{i}^{(\ell)}$ for simple.

Lemma 2.0.6
For every $\ell \in \mathbb{N}$ and non-negative $f \in \mathcal{H}_c$, there exists $j_f^{(\ell)} \in \mathcal{W}^\ell$ such that

$$\delta^{(\ell)} [\![f]\!]_1 \leq \left\| \hat{f}(*) - \hat{f}(*\ \psi_\ell^{-1}(j_f^{(\ell)})) \right\|_{\ell^2(G)},$$

where

$$\delta^{(\ell)} = \frac{\mathrm{i}^{(\ell)}}{2}.$$

Proof. This proof mainly consists of two steps. (*Step* 1) Define

$$\Gamma(f) := \sum_{\substack{j\in\mathcal{W}^\ell \\ g\in G}} \mu([j]) \left| \hat{f}^2(g) - \hat{f}^2(gg_{\widehat{j}}) \right|,$$

where $g_{(i_0,i_1,\ldots,i_{n-1})} := g_{i_0}g_{i_1}\cdots g_{i_{n-1}}$ for each $n\in\mathbb{N}$ and $(i_0,i_1,\ldots,i_{n-1})\in\mathcal{W}^n$, and for $j\in\mathcal{W}^\ell$ we set $\widehat{j} := (-i_{n-1},\ldots,-i_1,-i_0)$. Then, by the Cauchy-Schwarz's inequality we have

$$\begin{aligned}
\Gamma^2(f) &= \left\{ \sum_{\substack{j\in\mathcal{W}^\ell \\ g\in G}} \mu([j]) \left| \hat{f}^2(g) - \hat{f}^2(gg_{\widehat{j}}) \right| \right\}^2 \\
&= \left[\sum_{\substack{j\in\mathcal{W}^\ell \\ g\in G}} \left\{ \sqrt{\mu([j])} \left(\hat{f}(g) + \hat{f}(gg_{\widehat{j}}) \right) \right\} \left\{ \sqrt{\mu([j])} \left| \hat{f}(g) - \hat{f}(gg_{\widehat{j}}) \right| \right\} \right]^2 \\
&\leq \left\{ \sum_{\substack{j\in\mathcal{W}^\ell \\ g\in G}} \mu([j]) \left(\hat{f}(g) + \hat{f}(gg_{\widehat{j}}) \right)^2 \right\} \sum_{\substack{j\in\mathcal{W}^\ell \\ g\in G}} \mu([j]) \left| \hat{f}(g) - \hat{f}(gg_{\widehat{j}}) \right|^2. \qquad (2.0.3)
\end{aligned}$$

Here,

$$\begin{aligned}
\sum_{\substack{j\in\mathcal{W}^\ell \\ g\in G}} \mu([j]) \left(\hat{f}(g) + \hat{f}(gg_{\widehat{j}}) \right)^2 &= \sum_{\substack{j\in\mathcal{W}^\ell \\ g\in G}} \mu([j]) \hat{f}^2(g) + \sum_{\substack{j\in\mathcal{W}^\ell \\ g\in G}} \mu([j]) \hat{f}^2(gg_{\widehat{j}}) \\
&\quad + 2 \sum_{\substack{j\in\mathcal{W}^\ell \\ g\in G}} \mu([j]) \hat{f}(g) \hat{f}(gg_{\widehat{j}}) \\
&= \sum_{g\in G} \hat{f}^2(g) + \sum_{j\in\mathcal{W}^\ell} \mu([j]) \sum_{g\in G} \hat{f}^2(gg_{\widehat{j}}) \\
&\quad + 2 \sum_{j\in\mathcal{W}^\ell} \mu([j]) \sum_{g\in G} \hat{f}(g) \hat{f}(gg_{\widehat{j}}) \\
&\leq 4 [\![f]\!]_1^2,
\end{aligned}$$

where the last inequality is obtained by Cauchy-Schwarz's inequality and (2.0.1). We consequently have

$$\begin{aligned}
\Gamma^2(f) &\leq 4[\![f]\!]_1^2\left\{\sum_{j\in\mathcal{W}^\ell}\mu([j])\left(\sum_{g\in G}\left|\hat{f}(g)-\hat{f}(gg_{\widehat{j}})\right|^2\right)\right\}\\
&= 4[\![f]\!]_1^2\left\{\sum_{j\in\mathcal{W}^\ell}\mu([j])\left\|\hat{f}(*)-\hat{f}(*\,\psi_\ell^{-1}(j))\right\|^2_{\ell^2(G)}\right\}.
\end{aligned}\tag{2.0.4}$$

Since μ is a probability, there exists $j_f^{(\ell)}\in\mathcal{W}^\ell$ such that

$$4[\![f]\!]_1^2\left\{\sum_{j\in\mathcal{W}^\ell}\mu([j])\left\|\hat{f}(*)-\hat{f}(*\,\psi_\ell^{-1}(j))\right\|^2_{\ell^2(G)}\right\}\leq 4[\![f]\!]_1^2\left\|\hat{f}(*)-\hat{f}(*\,\psi_\ell^{-1}(j_f^{(\ell)}))\right\|^2_{\ell^2(G)},$$

It follows that

$$\Gamma(f)\leq 2[\![f]\!]_1\left\|\hat{f}(*)-\hat{f}(*\,\psi_\ell^{-1}(j_f^{(\ell)}))\right\|_{\ell^2(G)}.\tag{2.0.5}$$

(*Step* 2) We want to find a constant C such that for every positive $f\in\mathcal{H}_c$

$$C[\![f]\!]_1^2\leq\Gamma(f).$$

In order to find it, so we manipulate $\Gamma(f)$ in following way : If $gg_{\widehat{j}}=h$ for $g,h\in G$ and $j\in\mathcal{W}^\ell$, then $\hat{f}(gg_{\widehat{j}})=\hat{f}(h)$ and $\hat{f}(g)=\hat{f}(hg_{\widehat{j}}^{-1})=\hat{f}(hg_j)$. This implies that

$$\left|\hat{f}^2(g)-\hat{f}^2(gg_{\widehat{j}})\right|=\left|\hat{f}^2(hg_j)-\hat{f}^2(h)\right|=\left|\hat{f}^2(h)-\hat{f}^2(hg_j)\right|.\tag{2.0.6}$$

Consider a partition E and $\widehat{E}$ of $\mathcal{W}^\ell$ so that $j\in E$ if and only if $\widehat{j}\in\widehat{E}$. Clearly, since we can construct a bijection γ, $\sharp E=\sharp\widehat{E}=\sharp\mathcal{W}^\ell/2$. By the aboves, we have

$$\begin{aligned}
\Gamma(f) &= \sum_{\substack{j\in\mathcal{W}^\ell\\ g\in G}}\mu([j])\left|\hat{f}^2(g)-\hat{f}^2(gg_{\widehat{j}})\right|\\
&=\sum_{j\in E}\mu([j])\sum_{g\in G}\left|\hat{f}^2(g)-\hat{f}^2(gg_{\widehat{j}})\right|+\sum_{\widehat{j}\in\widehat{E}}\mu([\widehat{j}])\sum_{h\in G}\left|\hat{f}^2(h)-\hat{f}^2(hg_j)\right|\\
&=\sum_{j\in E}\mu([j])\sum_{g\in G}\left|\hat{f}^2(g)-\hat{f}^2(gg_{\widehat{j}})\right|+\sum_{\widehat{j}\in\widehat{E}}\mu([\widehat{j}])\sum_{g\in G}\left|\hat{f}^2(g)-\hat{f}^2(gg_{\widehat{j}})\right|\\
&=\sum_{j\in E}\left\{\mu([j])+\mu([\widehat{j}])\right\}\sum_{g\in G}\left|\hat{f}^2(g)-\hat{f}^2(gg_{\widehat{j}})\right|.
\end{aligned}\tag{2.0.7}$$

Since $M_\ell(j)=M_\ell\left(\widehat{j}\right)$, we therefore have

$$\Gamma(f)=\frac{1}{2}\sum_{\substack{j\in\mathcal{W}^\ell\\ g\in G}}M_\ell(j)\left|\hat{f}^2(g)-\hat{f}^2(gg_{\widehat{j}})\right|.$$

Then we may have by [Woe00, Proposition 4.3]

$$\mathfrak{i}^{(\ell)} [\![f]\!]_1^2 \leq \Gamma(f). \tag{2.0.8}$$

(*Step 3*) If we combine (2.0.5) with (2.0.8), then we have

$$\frac{\mathfrak{i}^{(\ell)}}{2} [\![f]\!]_1 \leq \left\| \hat{f}(*) - \hat{f}(* \ \psi_\ell^{-1}(j_f^{(\ell)})) \right\|_{\ell^2(G)}.$$

□

The following lemma follows from the arguments in [Sta13, Lemma 5.3]. The novelty is that we carefully investigate the constants involved in [Sta13, Lemma 5.2].

Lemma 2.0.7
Suppose that N has an element of word length ℓ. Let $\varphi(\omega) = \varphi(\omega_1, w_2)$ with $L_\varphi \mathbf{1} = \mathbf{1}$, and invariant Gibbs measure $\mu = \mu_\varphi$. Put $k = \ell + 4$. Suppose that there exists $\delta^{(k)} > 0$ such that, for $f \in \mathcal{H}_c$, ${}^\exists j_f^{(k)} \in \mathcal{W}^k$ such that

$$\delta^{(k)} [\![f]\!]_1 \leq \left\| \hat{f}(*) - \hat{f}(* \ \psi_\ell^{-1}(j_f^{(k)})) \right\|_{\ell^2(G)}. \tag{$\spadesuit$}$$

Then there exists $\delta, \alpha > 0$ (see (2.0.11) and below), such that ${}^\forall n \in \mathbb{N}$

$$[\![\mathcal{L}_\varphi^{3kn}(f)]\!]_1 \leq (1 - \alpha\delta)^n [\![f]\!]_1,$$

where

$$\alpha = \inf \left\{ \Phi_{3k}\left(\tau_j(x)\right) \middle| x \in \theta^{3k}([j]), j \in \mathcal{W}^\dagger \right\}, \quad \delta = 1 - \sqrt{1 - \left(\frac{\delta^{(k)}}{2}\right)^2}.$$

Proof. We find standard elements u and loops v of length k by using $j_f^{(k)}$ and $\mathcal{J} \subset \mathcal{W}^k$ as in the previous Lemma. More precisely, there exists finite subset $\mathcal{W}^\dagger$ of $\mathcal{W}^{3k}$ such that for $i = 1, 2$ ${}^\forall n_i \in \mathbb{N}$ and ${}^\forall w_i \in \mathcal{W}^{n_i}$, ${}^\exists u = u(w_1, f, w_2)$ and $v = v(w_1, f, w_2) \in \mathcal{W}^\dagger$ such that all of the followings hold;

1. $w_1 u(w_1, f, w_2) w_2$ and $w_1 v(w_1, f, w_2) w_2$ are admissible,
2. $\psi_{3k}\left(u(w_1, f, w_2)\right) = \psi_k(j_f^{(k)})$ and $\psi_{3k}(v(w_1, f, w_2)) = e$,
3. the first k letters of $u(w_1, f, w_2)$ and $v(w_1, f, w_2)$ coincide.

Actually, we first connect an arbitrary element w_1 with $j_f^{(k)}$, and $j_f^{(k)}$ with w_2 by certain elements of $\mathcal{J}$, say j', j'', respectively. For j', j'', we can find a element $w_{j',j''} \in \mathcal{J}$ as in Lemma 2.0.6. By combining the above items, for w_1 and w_2, the elements

$$u := j' j_f^{(k)} j'' \qquad v := j' w_{j',j''} j''$$

satisfy the above three conditions.

Since assumption (♠) holds and

$$\|f \circ \tau_u(x,*) + f \circ \tau_v(x,*)\|_{\ell^2(G)} = \left\| \hat{f}(* \ \psi_k^{-1}(j_f^{(k)})) + \hat{f}(*) \right\|_{\ell^2(G)},$$

we have by (2.0.2) for all $x \in [a]$

$$\frac{1}{2} \|f \circ \tau_u(x,*) + f \circ \tau_v(x,*)\|_{\ell^2(G)} \leq (1-\delta)[\![f]\!]_1,$$

where $a \in \mathcal{W}$ satisfies that $w_2 a$ is admissible and

$$\delta = 1 - \sqrt{1 - \left(\frac{\delta^{(k)}}{2}\right)^2}.$$

Next, $^\forall N \in \mathbb{N}$ and $^\forall w = w_1 w_2$ of length N and $|w_i| = n_i$ for $i = 1, 2$,

$$\begin{aligned} \Phi_N(\tau_w(x)) = \Phi_N(\tau_{w_1 w_2}(x)) &= \prod_{k=0}^{n_1-1} \varphi(\theta^k(w_1 w_2 x)) \prod_{k=n_1}^{n_1+n_2-1} \varphi(\theta^k(w_1 w_2 x)) \\ &= \prod_{k=0}^{n_1-1} \varphi(\theta^k(w_1 w_2 x)) \prod_{k=0}^{n_2-1} \varphi(\theta^k(w_2 x)) \\ &= \Phi_{n_1}(\tau_{w_1 w_2}(x)) \Phi_{n_2}(\tau_{w_2}(x)). \end{aligned} \tag{2.0.9}$$

For all $n \in \mathbb{N}$ and $w \in \mathcal{W}^n$, since the normalized potential φ depends only on the first two coordinates,

$$\frac{\Phi_n \circ \tau_w \left(\tau_{u(w,f,x_0)}(x)\right)}{\Phi_n \circ \tau_w \left(\tau_{v(w,f,x_0)}(x)\right)} = 1 \tag{2.0.10}$$

where x_0 is the beginning word of x. Let

$$\alpha := \inf \left\{ \Phi_{3k}\left(\tau_j(x)\right) \middle| x \in \theta^{3k}([j]), j \in \mathcal{W}^\dagger \right\}.$$

By dividing each $u \in \mathcal{W}^\dagger$ into two words u_1 and u_2 and setting $\Phi_{3k}\left(\tau_{u_2}(x)\right) := \Phi_{3k}\left(\tau_u(x)\right) - \alpha/2$ and $\Phi_{3k}\left(\tau_{u_1}(x)\right) := \alpha/2$ for each $x \in \theta^{3k}([u])$, we may assume without loss of generality that $\Phi_{3k}\left(\tau_u(x)\right) = \alpha/2$ for all $x \in \theta^{3k}([u])$ and $u \in \mathcal{W}^\dagger$.

For $i = 1, 2, \dots n$, $w_i \in \mathcal{W}^{n_i}$ with $|w_i| = n_i$. For a finite word w, set $f_w(x,g) := f \circ \tau_w(x,g) = f\left(\tau_w(x), g\psi(w)^{-1}\right)$, and define by induction, for $j = 1, 2, \dots, p$,

$$u_j^{(1)} := u(w_{j-1}, f_j, w_j), \qquad u_j^{(2)} := v(w_{j-1}, f_j, w_j).$$

For $N := n_0 + n_1 + \cdots + n_n + 3kn$ and $i = 1, 2$, we have

$$\begin{aligned} &\left\| \sum_{(i_1,\dots i_n) \in \{1,2\}^n} \Phi_N\left(\tau_{w_0 u_1^{(i_1)} w_1 \cdots u_n^{(i_n)} w_n}(x)\right) f_{w_0 u^{(i_1)} w_1 \cdots u_n^{(i_n)} w_n}(x, \cdot) \right\|_{\ell^2(G)} \\ &\leq \left(\max_{(i_1,\dots i_n) \in \{1,2\}^n} \Phi_N\left(\tau_{w_0 u_1^{(i_1)} w_1 \cdots u_n^{(i_n)} w_n}(x)\right)\right) \{2(1-\delta)\}^n [\![f]\!]_1. \end{aligned}$$

Without loss of generality, we may assume that the maximizing index i_j is equal to 1 for all j. Then by (2.0.10) and (2.0.9)

$$\begin{aligned}
& 2\Phi_N\left(\tau_{w_0u_1^{(1)}w_1\cdots u_n^{(1)}w_n}(x)\right) \\
= \; & 2\Phi_{n_0}\left(\tau_{w_0u_1^{(1)}}(x)\right)\Phi_{N-n_0}\left(\tau_{u_1^{(1)}w_1\cdots u_n^{(1)}w_n}(x)\right) \\
= \; & \left\{\Phi_{n_0}\circ\tau_{w_0}\left(\tau_{u_1^{(1)}}(w_1\cdots w_nx)\right)+\Phi_{n_0}\circ\tau_{w_0}\left(\tau_{u_1^{(2)}}(w_1\cdots w_nx)\right)\right\} \\
& \times\Phi_k\left(\tau_{u_1^{(1)}}(w_1\cdots w_nx)\right)\Phi_{N-(n_0+k)}\left(\tau_{w_1\cdots u_n^{(1)}w_n}(x)\right) \\
= \; & \Phi_{n_0}\circ\tau_{w_0}\left(\tau_{u_1^{(1)}}(w_1\cdots w_nx)\right)(\alpha/2)\Phi_{N-(n_0+k)}\left(\tau_{w_1\cdots u_n^{(1)}w_n}(x)\right) \\
& +\Phi_{n_0}\circ\tau_{w_0}\left(\tau_{u_1^{(2)}}(w_1\cdots w_nx)\right)(\alpha/2)\Phi_{N-(n_0+k)}\left(\tau_{w_1\cdots u_n^{(1)}w_n}(x)\right) \\
\leq \; & \Phi_{n_0}\circ\tau_{w_0}\left(\tau_{u_1^{(1)}}(w_1\cdots w_nx)\right)\Phi_k\left(\tau_{u_1^{(1)}}(w_1\cdots w_nx)\right)\Phi_{N-(n_0+k)}\left(\tau_{w_1\cdots u_n^{(1)}w_n}(x)\right) \\
& +\Phi_{n_0}\circ\tau_{w_0}\left(\tau_{u_1^{(2)}}(w_1\cdots w_nx)\right)\Phi_k\left(\tau_{u_1^{(2)}}(w_1\cdots w_nx)\right)\Phi_{N-(n_0+k)}\left(\tau_{w_1\cdots u_n^{(1)}w_n}(x)\right) \\
= \; & \Phi_{n_0+k}\left(\tau_{w_0u_1^{(1)}w_1\cdots w_n}(x)\right)\Phi_{N-(n_0+k)}\left(\tau_{w_1u_2^{(1)}\cdots u_n^{(1)}w_n}(x)\right) \\
& +\Phi_{n_0+k}\left(\tau_{w_0u_1^{(2)}w_1\cdots w_n}(x)\right)\Phi_{N-(n_0+k)}\left(\tau_{w_1u_2^{(1)}\cdots u_n^{(1)}w_n}(x)\right)
\end{aligned}$$

Therefore, by an inductive calculation and suitable replace of indices,

$$2^n\Phi_N\left(\tau_{w_0u_1^{(1)}w_1\cdots u_n^{(1)}w_n}(x)\right)\leq\sum_{(i_1,\dots i_n)\in\{1,2\}^n}\Phi_N\left(\tau_{w_0u_1^{(i_1)}w_1\cdots u_n^{(i_n)}w_n}(x)\right),$$

So we conclude that for $i=1,2$

$$\begin{aligned}
& \left\|\sum_{(i_1,\dots i_n)\in\{1,2\}^n}\Phi_N\left(\tau_{w_0u_1^{(i_1)}w_1\cdots u_n^{(i_n)}w_n}(x)\right)f_{w_0u^{(i_1)}w_1\cdots u_n^{(i_n)}w_n}(x,\cdot)\right\|_{\ell^2(G)} \\
\leq & \left\{\sum_{(i_1,\dots i_n)\in\{1,2\}^n}\Phi_N\left(\tau_{w_0u_1^{(i_1)}w_1\cdots u_n^{(i_n)}w_n}(x)\right)\right\}(1-\delta)^n[\![f]\!]_1.
\end{aligned}$$

It follows from the arguments [Sta13, Step3 in Lemma 5.3] that

$$[\![\mathcal{L}_\varphi^{3kn}(f)]\!]_1\leq(1-\alpha\delta)^n[\![f]\!]_1, \tag{2.0.11}$$

□

Combining Lemma 2.0.6, Lemma 2.0.7 with [Sta13, Lemma 5.2] (see also [Jae15, Lemma 3.2]), we obtain our main result.

Theorem 2.0.8
Suppose that $G=\mathbb{F}_2/N$ is non-amenable and N has an element of word length ℓ. Put $k=\ell+4$, then

$$\mathcal{P}(\widetilde{\Sigma}_A,T,\varphi)\leq\frac{1}{3k}\log(1-\alpha\delta),$$

where

$$\alpha = \inf \left\{ \Phi_{3k}\left(\tau_j(x)\right) \middle| x \in \theta^{3k}([j]), j \in \mathcal{W}^{\dagger} \right\},$$

and

$$\delta = 1 - \sqrt{1 - \left(\frac{\delta^{(k)}}{2}\right)^2} = 1 - \sqrt{1 - \left(\frac{\mathfrak{i}^{(k)}}{4}\right)^2}.$$

In view of the next example, it would be interesting to derive a sharp estimate.

Example 2.0.9

Consider that $G = \mathbb{F}_3/N$ where $\mathbb{F}_3 = \langle g_1, g_2, g_3 \rangle$ and N is the smallest normal subgroup containing g_3. Since N has an element of length 1 and Remark 2.0.2, $k = 3$. Define the constant potential $\varphi \equiv \frac{1}{5}$. Then we have $L_\varphi(\mathbf{1}) = \mathbf{1}$, $\mu([j]) = \left(\frac{1}{6 \cdot 5^2}\right)^3$ for any $j \in \mathcal{W}^3$ and $\alpha = \left(\frac{1}{5}\right)^9$. In fact, we can estimate that $\mathfrak{i}^{(3)} = \frac{103}{6 \cdot 5^2}$. Hence, if we apply the first approximation $\log(1-x) \leq x$ for $0 < x < 1$, then we have

$$\mathcal{P}(\widetilde{\Sigma}_A, T, \varphi) \leq \frac{1}{3k} \log(1 - \alpha\delta) \leq 8.45 \times 10^{-10}$$

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